4.9 Gershgorin Circle Theorem

Monday, March 8, 2021 3:25 PM

Thm 4.5 (Gershgorin) Let $A \in \mathbb{C}^{n \times n}$.

Let $D_i = \{ |z - a_{ii}| \le r_i \mid z \in \mathbb{C} \}$, where $r_i = \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|$.

Then all eigenvalues λ_i of A lie in $\bigcup_{i=1}^{n} D_i$

In particular, if λ is an eigenvalue of A, then for some i,

$$|\lambda - a_{ii}| \le r_i.$$

proof. Let (λ, v) be an eigenpair of A, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$.

$Av = \lambda v$, so $\lambda v_i = \sum_{j=1}^{n} a_{ij} v_j$.

$$\Rightarrow (\lambda - a_{ii}) v_i = \sum_{\substack{j=1 \\ j \ne i}}^{n} a_{ij} v_j.$$

Let v_k be the element with greatest magnitude, i.e. $|v_k| \ge |v_j| \ \forall j = 1, \dots, n$.

Then $\left| \frac{v_j}{v_k} \right| \le 1$.

$$\Rightarrow |\lambda - a_{kk}| \le \sum_{\substack{j=1 \\ j \ne k}}^{n} |a_{kj}| \left| \frac{v_j}{v_k} \right| \le \sum_{\substack{j=1 \\ j \ne k}}^{n} |a_{kj}|, \text{ so } \lambda \in D_k$$

$\blacksquare$

Corollary 4.3 Let $A \in \mathbb{R}^{n \times n}$. If

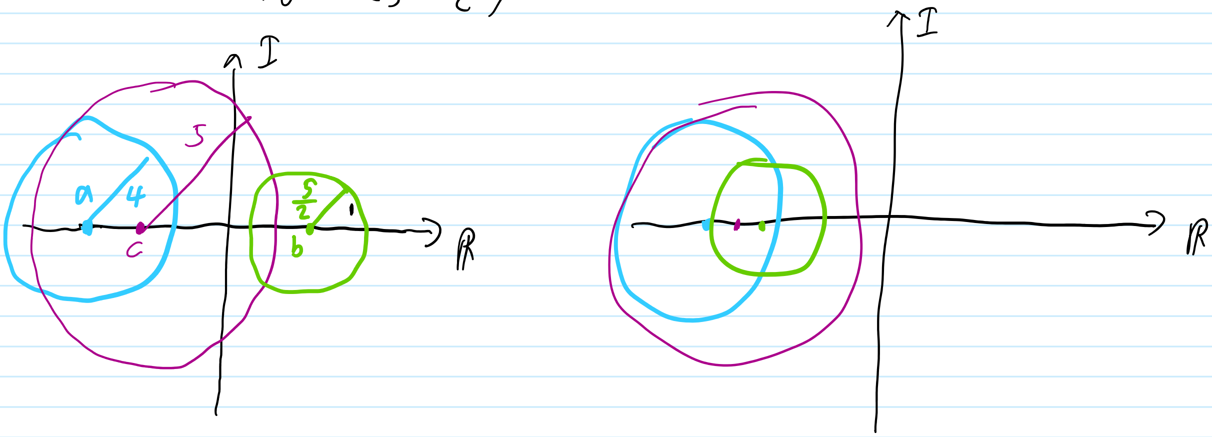
$$a_{ii} < -r_i, \text{ where } r_i = \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| \text{ for } i = 1, 2, \dots, n,$$

then the eigenvalues of A have negative real parts.

Ex.

$$\begin{matrix} & & & (|-1|+|3|) & |\tfrac{1}{2}|+|-2| & |-5| \end{matrix}$$

$$A = \begin{pmatrix} a & -1 & 3 \\ \frac{1}{2} & b & -2 \\ 0 & -5 & c \end{pmatrix} \quad \text{Then} \quad r_1 = 4, \quad r_2 = \frac{5}{2}, \quad r_3 = 5$$



So if $a < -4$, $b < -\frac{5}{2}$, $c < -5$, then all eigenvalues have negative real parts.

Ex. 4.16

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}. \quad \text{Then} \quad r_1 = r_3 = 1 \text{ and } r_2 = 2.$$

but $a_{22} = -r_2$.

So Corollary 4.3 does not say that eigenvalues have neg. real part.

But $\lambda_{1,2,3} = -2,\ -2 \pm \sqrt{2}$, so eigenvalues do have neg real part.

(i.e. Corollary 4.3 is sufficient but not necessary)

Ex.